



PHYSICS BY LAMHI

Lamhi's Handwritten Notes

Electrostatics & Current Electricity

Transcribed and verified from Lamhi's own class notebook — every formula and worked problem checked line by line, with the same rigor as the site's detailed Notes pages.

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Electric Charge

Electric charge is a scalar quantity. Electric charge is conserved — charge can neither be created nor destroyed, only transferred from one body to another.

- S.I. unit — coulomb (C)
- C.G.S. unit — stat-coulomb (esu)
- $1 \text{ C} = 3 \times 10^9 \text{ stat-coulomb}$

Quantisation of Charge

Charge transferred from one body to another is always in discrete form — an integral multiple of the minimum amount of charge.

QUANTISATION OF CHARGE

$$q = \pm ne, \quad n = 1, 2, 3, \dots$$

Here $e = 1.6 \times 10^{-19} \text{ C}$ is the elementary charge and n is the number of charges transferred, $n = \frac{q}{e}$.

Charge also shows an **additive property**, i.e. individual charges are added or subtracted algebraically to get the total charge.

Coulomb's Law

The force of attraction or repulsion between two point charges in air/vacuum is directly proportional to the product of the charges and inversely proportional to the square of the distance between them, and acts along the line joining the two charges.

$$F \propto q_1 q_2, \quad F \propto \frac{1}{r^2} \Rightarrow F \propto \frac{q_1 q_2}{r^2}$$

COULOMB'S LAW

$$F = \frac{k q_1 q_2}{r^2}$$

where $k = \frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \frac{\text{N m}^2}{\text{C}^2}$ and ϵ_0 is the permittivity of free space, $\epsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N m}^2}$.

For air/vacuum:

$$F_a = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r^2}$$

For any other medium:

$$F_m = \frac{1}{4\pi\epsilon} \cdot \frac{q_1 q_2}{r^2} = \frac{1}{4\pi K \epsilon_0} \cdot \frac{q_1 q_2}{r^2}, \quad \epsilon = K \epsilon_0$$

where K is the dielectric constant (relative permittivity) of the medium. Comparing the two forces,

EFFECT OF MEDIUM

$$\frac{F_m}{F_a} = \frac{1}{K} \Rightarrow K F_m = F_a$$

Validity of Coulomb's Law

- Coulomb's law is obeyed only for point charges, not for a distribution of charges.
- Coulomb's law holds only for static (stationary) charges.
- The separation between the charges must be greater than 10^{-15} m (nuclear size), because at separations of 10^{-15} m the strong nuclear force dominates over the electrostatic force.

Coulomb's Law in Vector Form

Consider two point charges q_1 and q_2 at positions with position vectors $\vec{r}_1$ and $\vec{r}_2$ from an origin O . By the triangle law, the vector from charge 1 to charge 2 is $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$, so $\hat{r}_{12} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$ (and likewise $\hat{r}_{21} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$).

Force on q_1 due to q_2 :

$$\vec{F}_{12} = \frac{k q_1 q_2}{|\vec{r}_{21}|^2} \hat{r}_{21} = \frac{k q_1 q_2 (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$$

Force on q_2 due to q_1 :

$$\vec{F}_{21} = \frac{k q_1 q_2}{|\vec{r}_{12}|^2} \hat{r}_{12} = \frac{k q_1 q_2 (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

so that, as required by Newton's third law,

$$\vec{F}_{12} = -\vec{F}_{21}, \quad \hat{r}_{12} = -\hat{r}_{21}$$

Superposition Principle

According to the superposition principle, the force acting on a test charge due to a group of charges is equal to the vector sum of the forces due to each individual charge, acting alone.

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n, \quad \vec{F}_i = \frac{k q_0 q_i}{r_i^2} \hat{r}_i$$

Worked problem

Four point charges $q_A = 2 \mu\text{C}$, $q_B = -5 \mu\text{C}$, $q_C = 2 \mu\text{C}$, $q_D = -5 \mu\text{C}$ are located at the corners of a square $ABCD$ of side 10 cm. What is the force on a charge of $1 \mu\text{C}$ placed at the centre O of the square?

The centre is equidistant from all four corners: splitting each side at its midpoint gives two legs of 5 cm each, so

$$OA = OB = OC = OD = \sqrt{5^2 + 5^2} = 5\sqrt{2} \text{ cm}$$

A and C (opposite corners, equal charge) are equidistant from O along the same diagonal, so the forces they exert on the centre charge are equal in magnitude and opposite in direction, $\vec{F}_{OA} = -\vec{F}_{OC}$. The same holds for B and D , $\vec{F}_{OB} = -\vec{F}_{OD}$.

$$\vec{F}_O = \vec{F}_{OA} + \vec{F}_{OC} + \vec{F}_{OB} + \vec{F}_{OD} = 0$$

Answer: The net force on the $1 \mu\text{C}$ charge at the centre is zero.

Electric Field

The electric field is the space around a charged body in which a test charge experiences a force of attraction or repulsion.

Electric field intensity is the force acting per unit test charge:

$$\vec{E} = \frac{\vec{F}}{q_0} \text{ (N/C)}$$

For a test charge q_0 placed at distance r from a source charge q , the force on it is $\vec{F} = \frac{k q q_0}{r^2} \hat{r}$, so

$$\frac{\vec{F}}{q_0} = \frac{k q}{r^2} \hat{r} \quad \Rightarrow \quad \vec{E} = \frac{k q}{r^2} \hat{r}$$

Plotting field intensity against distance from the charge gives a curve that falls off sharply and flattens out at large r — the inverse-square fall-off of E with r .

Electric Field Lines

Field lines point radially outward from a positive charge and radially inward toward a negative charge.

- Electric field lines are, in general, discontinuous — but the field line directly between two point charges is continuous.
- Electric field lines never intersect each other — if they did, the field at that point would have two directions, which is not possible.
- Electric field lines are crowded where the field is strong and wider apart where the field is weak.
- Electric field lines never form closed loops.
- Electric field lines are outward or inward normal to a charged body (the surface of a conductor).
- When two charges Q_1 and Q_2 are some distance apart, their field lines may appear to attract or repel each other depending on the nature (like/unlike) of the charges.

Distribution of Charges

When charge is distributed over the length, area, or volume of a conductor, it is called a **continuous charge distribution**.

- **Linear charge distribution** — charge per unit length of a wire, $\lambda = \frac{q}{l}$
- **Surface charge distribution** — charge per unit area of a conductor, $\sigma = \frac{q}{A}$
- **Volume charge distribution** — charge per unit volume of a conductor, $\rho = \frac{q}{V}$

Worked problem

Two point charges $q_A = 3 \mu\text{C}$ and $q_B = -3 \mu\text{C}$ are located 20 cm apart in vacuum. (i) What is the electric field at the midpoint O of the line AB joining the two charges? (ii) If a negative test charge of magnitude $1.5 \times 10^{-9} \text{ C}$ is placed at this point, what is the force experienced by the test charge?

O is 10 cm from each charge. The field due to $+3 \mu\text{C}$ points away from A (toward B); the field due to $-3 \mu\text{C}$ points toward B as well — so the two fields add:

$$E = E_1 + E_2 = \left[\frac{(9 \times 10^9)(3 \times 10^{-6})}{(0.10)^2} \right] \times 2 = 5.4 \times 10^6 \text{ N/C, along } OB$$

For the negative test charge $q = -1.5 \times 10^{-9} \text{ C}$ placed at O , the force $\vec{F} = q\vec{E}$ points opposite to $\vec{E}$ — i.e. along OA (it is pulled toward the positive charge and pushed away from the negative charge, both toward A):

$$F = (1.5 \times 10^{-9})(5.4 \times 10^6) = 8.1 \times 10^{-3} \text{ N, along } OA$$

Answer: $E = 5.4 \times 10^6 \text{ N/C}$ along OB ; $F = 8.1 \times 10^{-3} \text{ N}$ along OA .

Worked problem

A polythene piece rubbed with wool is found to have a negative charge of $3 \times 10^{-7} \text{ C}$. (i) Estimate the number of electrons transferred (from which body to which). (ii) Is there a transfer of mass from wool to polythene?

Since wool becomes positively charged and polythene negatively charged, electrons are transferred from wool to polythene.

$$q = ne \Rightarrow n = \frac{q}{e} = \frac{3 \times 10^{-7}}{1.6 \times 10^{-19}} \approx 1.87 \times 10^{12}$$

Mass is transferred too, since each electron carries mass $m_e = 9.11 \times 10^{-31}$ kg:

$$\text{Total mass transferred} = m_e \times n \approx (9.11 \times 10^{-31})(1.87 \times 10^{12}) \approx 1.70 \times 10^{-18} \text{ kg}$$

Answer: About 1.87×10^{12} electrons are transferred from wool to polythene. Yes, a mass of about 1.70×10^{-18} kg is transferred along with them — though it is far too small to be of any practical consequence.

Worked problem

Two identical conducting spheres *A* and *B*, with centres 50 cm apart, each carry a charge of 6.5×10^{-7} C. A third identical but uncharged sphere is touched first to *A*, then to *B*, and finally removed. What is the new force of repulsion between *A* and *B*?

Touching the uncharged sphere to *A* splits *A*'s charge equally: each now carries $q/2$. Touching that sphere (carrying $q/2$) to *B* (carrying q) shares their combined charge equally again, so each ends up with:

$$\frac{1}{2} \left(\frac{q}{2} + q \right) = \frac{3q}{4}$$

So *A* is left with $q/2$ and *B* with $3q/4$:

$$F = \frac{1}{4\pi\epsilon_0} \cdot \frac{(q/2)(3q/4)}{r^2} = \frac{1}{4\pi\epsilon_0} \cdot \frac{3q^2}{8r^2} = \frac{(9 \times 10^9)(3)(6.5 \times 10^{-7})^2}{(0.50)^2}$$

$$F \approx 5.70 \times 10^{-3} \text{ N}$$

Answer: The new force of repulsion is about 5.70×10^{-3} N.

Electric Dipole

If two equal and opposite charges are separated by some distance, they form an **electric dipole**. The strength of an electric dipole is called its **dipole moment** — the product of either charge and the distance of separation between them.

For charges $-q$ and $+q$ placed a distance a on either side of a centre *O* (total separation $2a$):

DIPOLE MOMENT

$$\vec{p} = q(2a)$$

The direction of the dipole moment is from the negative charge to the positive charge.

Electric Field Due to a Dipole on the Axial Line

Consider an electric dipole of moment $\vec{p}$ made of charges $-q$ at A and $+q$ at B, separated by $2a$. Point P lies on the axial line (the line through both charges), beyond the $+q$ end, at a distance r from the centre O of the dipole, so that $AP = r + a$ and $BP = r - a$.

Field at P due to the $-q$ charge (pointing from P towards A):

$$\vec{E}_1 = \frac{kq}{(AP)^2} = \frac{kq}{(r + a)^2}$$

Field at P due to the $+q$ charge (pointing away from B, same direction as $\vec{p}$):

$$\vec{E}_2 = \frac{kq}{(BP)^2} = \frac{kq}{(r - a)^2}$$

Since $BP < AP$, $E_2 > E_1$, and the two fields point in opposite directions, so the net field points along $\vec{p}$:

$$E = E_2 - E_1 = kq \left[\frac{1}{(r - a)^2} - \frac{1}{(r + a)^2} \right] = \frac{4kqar}{(r^2 - a^2)^2}$$

Writing $4kqar = 2kr(2qa) = 2kpr$ (since $p = 2aq$):

AXIAL FIELD (EXACT)

$$E_{\text{axial}} = \frac{2kpr}{(r^2 - a^2)^2}$$

For a point far from a short dipole ($r \gg a$), a^2 can be dropped next to r^2 , giving the usual short-dipole result:

$$E_{\text{axial}} \approx \frac{2kp}{r^3}$$

Electric Field Due to a Dipole on the Equatorial Line

Now let P lie on the equatorial line (perpendicular bisector of the dipole), at a distance r from the centre O. Both charges are then equidistant from P: $AP = BP = \sqrt{r^2 + a^2}$, so the two field magnitudes are equal:

$$E_1 = E_2 = \frac{kq}{r^2 + a^2}$$

Resolving each field into components along and perpendicular to the axis: the perpendicular components ($E_1 \sin \theta$ and $E_2 \sin \theta$) are equal and opposite, so they cancel; the components along the

axis ($E_1 \cos \theta$ and $E_2 \cos \theta$) point the same way and add, where $\cos \theta = \frac{a}{\sqrt{r^2 + a^2}}$.

EQUATORIAL FIELD

$$E_{\text{eq}} = 2E_1 \cos \theta = \frac{2kqa}{(r^2 + a^2)^{3/2}} = \frac{kp}{(r^2 + a^2)^{3/2}}$$

Note: when a dipole is placed in a *non-uniform* electric field, the force on each charge differs in magnitude at every point, so the dipole is never in stable equilibrium there — it always remains unstable.

Dipole in a Two-Dimensional (Uniform) Electric Field

Consider a dipole placed in a uniform field $\vec{E}$ making an angle θ with the field. Equal and opposite forces qE act on the two charges, so the net force is zero, but since they act at different points separated by a perpendicular arm of $2a \sin \theta$, they form a couple:

$$\tau = F_1(0) + F_2(2a \sin \theta) = qE(2a \sin \theta) = pE \sin \theta$$

TORQUE ON A DIPOLE

$$\vec{\tau} = \vec{p} \times \vec{E}$$

In a uniform field, $\vec{F}_{\text{net}} = 0$, but the torque rotates the dipole to align it with the field.

Electric Potential Energy of a Dipole

Since a torque acts on a dipole placed in a uniform field, work must be done to rotate it against that torque, and this work is stored as electric potential energy.

$$dW = \tau d\theta = pE \sin \theta d\theta$$

Total work done to rotate the dipole from θ_1 to θ_2 :

$$W = \int_{\theta_1}^{\theta_2} pE \sin \theta d\theta = pE(\cos \theta_1 - \cos \theta_2)$$

Taking the reference angle $\theta_1 = 90^\circ$ (so $U = 0$ there) gives the usual potential-energy expression $U(\theta) = -pE \cos \theta = -\vec{p} \cdot \vec{E}$.

Behaviour of a Charged Particle in a Uniform Electric Field

A charged particle q enters, with initial horizontal velocity u_x , a region between two parallel plates (bottom plate positive, top plate negative) where the field E points vertically upward, perpendicular to

its initial velocity. The particle deflects along the y -axis as it travels, tracing a parabolic path (just like projectile motion).

$$F = qE, \quad a_y = \frac{qE}{m}, \quad a_x = 0$$

Along x : $x = u_x t \implies t = \frac{x}{u_x}$. Along y (starting from rest in that direction):

$$y = \frac{1}{2} a_y t^2 = \frac{qE}{2m} \cdot \frac{x^2}{u_x^2} = K x^2, \quad K = \frac{qE}{2m u_x^2}$$

Electric Field Due to a Charged Body (Ring)

Consider a uniformly charged ring of radius a and linear charge density $\lambda = \frac{q}{2\pi a}$. Let P be a point on the axis of the ring at a distance x from its centre. Every charge element dq on the ring is the same distance from P: $r = \sqrt{a^2 + x^2}$.

Field at P due to an element dq :

$$dE = \frac{k dq}{a^2 + x^2} = \frac{k \lambda dl}{a^2 + x^2}$$

Each dE can be resolved into a component along the axis ($dE \cos \theta$) and one perpendicular to it ($dE \sin \theta$). By the ring's symmetry, every element has a counterpart elsewhere on the ring whose perpendicular component cancels it, so only the axial components survive and add up:

$$E = \int dE \cos \theta = \int \frac{k \lambda dl}{a^2 + x^2} \cdot \frac{x}{\sqrt{a^2 + x^2}}$$

Integrating dl around the full circumference $2\pi a$ (and using $\lambda \cdot 2\pi a = q$):

FIELD ON THE AXIS OF A RING

$$E = \frac{kqx}{(a^2 + x^2)^{3/2}}$$

Electric Potential

Electric potential at a point is the amount of work done in bringing a unit positive test charge from infinity to that point in the field, without acceleration.

$$V = \frac{W}{q_0} \quad (\text{unit: J/C, i.e. volt})$$

Potential is a scalar quantity — it may be positive or negative, but it has no direction.

Electric Potential Due to a Point Charge

Let P be a point at a distance r from a charge q . To find the potential at P, bring a unit test charge ($q_0 = 1$) from infinity up to P along the field line. Since the test charge moves *against* the field, the displacement and the field point in opposite directions:

$$dW = \vec{F} \cdot d\vec{x} = q_0 E dx \cos 180^\circ = -\frac{kq}{x^2} dx \quad (q_0 = 1)$$

$$W = - \int_{\infty}^r \frac{kq}{x^2} dx = \frac{kq}{r}$$

POTENTIAL DUE TO A POINT CHARGE

$$V = \frac{kq}{r}$$

Worked problem

A cube of side $\sqrt{3}$ m has a charge of $1 \mu\text{C}$ at each of its eight vertices. Find the potential and the electric field at the centre of the cube.

The centre O is equidistant from all eight vertices, at half the length of the cube's space diagonal. For a cube of side a , the space diagonal is found from the face diagonal ($\sqrt{2}a$) and an edge:

$$AD = \sqrt{(\sqrt{2}a)^2 + a^2} = \sqrt{3}a \implies AO = \frac{\sqrt{3}}{2}a$$

With $a = \sqrt{3}$ m, this gives $AO = 1.5$ m. Potential is a scalar, so the potential at O is just the sum of eight equal contributions:

$$V_0 = 8 \times \frac{kq}{AO} = \frac{16kq}{\sqrt{3}a} = \frac{16(9 \times 10^9)(10^{-6})}{3} = 48 \times 10^3 \text{ V}$$

The field, however, is a vector sum. Each vertex has a diametrically opposite vertex on the other end of a space diagonal through O, carrying an equal charge. The field contributions from any such pair at the centre are equal in magnitude and point in exactly opposite directions, so they cancel. Since all eight vertices pair up this way (four pairs in total), every contribution cancels and the net field at the centre is zero.

Answer: $V_0 = 48 \times 10^3 \text{ V} = 48 \text{ kV}$, and $\vec{E}_0 = 0$ (by symmetry).

Worked problem

Two charges, $4 \times 10^{-9} \text{ C}$ and $-3 \times 10^{-9} \text{ C}$, are located 0.1 m apart. At what point on the line joining the two charges is the electric potential zero? (Take the potential at infinity to be zero.)

Let x be the distance from $q_A = 4 \times 10^{-9} \text{ C}$ to the required point, so the point is $(0.1 - x)$ from $q_B = -3 \times 10^{-9} \text{ C}$. The point must lie between the charges for the two (oppositely signed) potentials to cancel:

$$V_A + V_B = 0 \implies \frac{kq_A}{x} + \frac{kq_B}{0.1 - x} = 0$$

$$\frac{4 \times 10^{-9}}{x} = \frac{3 \times 10^{-9}}{0.1 - x} \implies \frac{4}{x} = \frac{3}{0.1 - x}$$

$$0.4 - 4x = 3x \implies 0.4 = 7x \implies x = \frac{0.4}{7} \approx 0.057 \text{ m}$$

Answer: The potential is zero at a point about 0.057 m from the $4 \times 10^{-9} \text{ C}$ charge, between the two charges.

Electric Potential Energy of a System of Two Point Charges

The electric potential energy of a system of point charges is the total work done in bringing all the charges from infinity to their respective positions. Consider two charges, q_1 and q_2 , initially at infinity. Bringing q_1 to its position A costs no work, since there is no field yet. Bringing q_2 from infinity to position B, a distance r from A, against the potential set up by q_1 , costs:

$$W = q_2 V_1, \quad V_1 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1}{r}$$

This work done is stored as the potential energy of the two-charge system:

POTENTIAL ENERGY OF TWO CHARGES

$$U = \frac{kq_1 q_2}{r}$$

Conductor in an Electrostatic Field

- The net electric field intensity in the interior of a conductor is zero.
- The electric field just outside the surface of a charged conductor is perpendicular to that surface.
- The net charge in the interior of a conductor is zero.
- Charge resides only on the surface of the conductor.
- Electric potential is constant throughout the entire conductor.
- The surface charge distribution may be different at different points on the surface.

Dielectrics

Non-conducting materials that do not have free electrons are called dielectrics.

Polarisation of a Dielectric

Polarisation of a dielectric is the process of inducing equal and opposite charges on the two opposite faces of the dielectric on application of an external electric field.

Conservation Property of Electric Field

The work done in carrying a unit test charge between two points in an electric field is independent of the path taken — it depends only on the initial and final positions of the charge.

$$\oint \vec{E} \cdot d\vec{l} = \frac{W_{AB}}{q_0} = V_B - V_A$$

Note: if n identical liquid drops, each at potential V , merge to form one larger drop, both the total volume and the total charge are conserved, so the new radius is $R = n^{1/3}r$ and the new charge is $Q' = nq$.

$$V' = n^{2/3}V$$

Relation between Electric Field and Potential

The electric field at a point equals the negative rate of change of potential with distance:

$$E = -\frac{dV}{dx}$$

Electric Potential due to a Dipole at Any Point

Consider a dipole of moment $\vec{p}$ formed by charges $-Q$ and $+Q$ separated by a distance $2a$. A point P is at a distance r from the centre of the dipole, making an angle θ with the dipole axis.

Dropping perpendiculars from the two charges onto the line joining the centre to P gives, for $r \gg a$, the approximate distances from each charge to P :

$$r_1 = AP \approx r + a \cos \theta \quad (\text{distance from } -Q)$$

$$r_2 = BP \approx r - a \cos \theta \quad (\text{distance from } +Q)$$

Potential at P due to the $+Q$ charge of the dipole:

$$V_1 = \frac{KQ}{r_2} = \frac{KQ}{r - a \cos \theta}$$

Potential at P due to the $-Q$ charge of the dipole — since this charge is negative, its contribution to the potential is negative:

$$V_2 = -\frac{KQ}{r_1} = -\frac{KQ}{r + a \cos \theta}$$

Potential at P due to the dipole is the sum of these two:

$$V = V_1 + V_2 = \frac{KQ}{r - a \cos \theta} - \frac{KQ}{r + a \cos \theta}$$

$$V = KQ \left[\frac{(r + a \cos \theta) - (r - a \cos \theta)}{r^2 - a^2 \cos^2 \theta} \right] = \frac{2KQa \cos \theta}{r^2 - a^2 \cos^2 \theta}$$

Since the dipole moment is $p = Q(2a)$, this gives the standard result:

POTENTIAL DUE TO A DIPOLE

$$V = \frac{Kp \cos \theta}{r^2 - a^2 \cos^2 \theta}$$

Equipotential Surfaces

- An equipotential surface is a surface at every point of which the electric potential remains the same.
- The work done in carrying a test charge from one point to another on an equipotential surface is zero.
- The electric field is always perpendicular to an equipotential surface at every point.

For an isolated point charge, the equipotential surfaces are concentric circles (spheres, in three dimensions) centred on the charge, with the radial field lines cutting every circle at right angles. For a dipole, the surfaces are no longer circular — they pinch in close around each charge and distort into larger, rounder loops that enclose both charges as you move further away.

Electric Flux

Electric flux is the number of electric field lines passing through a given area. It is a scalar quantity, and it may be positive, negative, or zero.

Area vector: area by itself is a scalar, but an infinitesimally small area element is treated as a vector, directed along the outward normal to the surface.

$$\hat{n} = \frac{d\vec{S}}{dS}, \quad d\vec{S} = dS \hat{n}$$

Flux is the product of the electric field intensity and the component of the area vector along the direction of the field:

$$d\phi = \vec{E} \cdot d\vec{S} = E dS \cos \theta$$

- Area placed perpendicular to the field ($\theta = 0^\circ$): $d\phi = E dS$, and the total flux $\phi = EA$ is maximum.
- Area placed parallel to the field ($\theta = 90^\circ$): the flux $\phi = 0$ is minimum.

Note: flux is taken as outward (positive) when field lines leave a surface, and inward (negative) when they enter it.

Gauss's Law

The surface integral of the electric field intensity over any closed surface, in vacuum, equals $1/\epsilon_0$ times the total charge enclosed by that surface.

GAUSS'S LAW

$$\oint \vec{E} \cdot d\vec{S} = \frac{q_{in}}{\epsilon_0}$$

Proof: consider a charge q and draw a spherical Gaussian surface of radius r around it. At every point of this surface, the field intensity and the area vector point in the same (radially outward) direction:

$$\vec{E} = \frac{Kq}{r^2} \hat{r}, \quad \hat{n} = \frac{d\vec{S}}{dS}, \quad d\vec{S} = dS \hat{n}$$

$$\hat{r} \cdot \hat{n} = \cos 0^\circ = 1$$

$$\vec{E} \cdot d\vec{S} = \frac{Kq}{r^2} dS$$

Integrating both sides over the closed surface:

$$\oint \vec{E} \cdot d\vec{S} = \frac{Kq}{r^2} \oint dS = \frac{q}{4\pi\epsilon_0 r^2} (4\pi r^2) = \frac{q_{in}}{\epsilon_0}$$

Note: Gauss's law holds for any closed surface, but it can only be used to solve for E on surfaces with enough symmetry (spherical, cylindrical, or planar). On an irregular surface the field intensity is not constant over the surface, so the integral cannot be reduced to a simple product of E and the area.

Gauss's law can be used to quickly find the electric field due to several standard, highly symmetric charge distributions:

Electric Field due to a Long Line Charge

Consider a long line charge of length l with a charge q uniformly distributed along it, so that $q = \lambda l$. Draw a cylindrical Gaussian surface of radius r , coaxial with the wire.

At the curved surface of the cylinder, the field and the area vector point in the same direction; at the two flat end faces, they are perpendicular to each other:

$$d\phi_{\text{edge}} = E dS \cos 90^\circ + E dS \cos 90^\circ = 0$$

$$d\phi_{\text{curved}} = E dS \cos 0^\circ = E dS$$

Applying Gauss's law over the whole closed surface:

$$\oint \vec{E} \cdot d\vec{S} = \frac{q_{\text{in}}}{\epsilon_0} \Rightarrow E(2\pi r l) = \frac{\lambda l}{\epsilon_0}$$

FIELD DUE TO A LONG LINE CHARGE

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = \frac{2K\lambda}{r}$$

Electric Field due to a Uniformly Charged Spherical Shell

Let the shell have radius R and a uniform surface charge density:

$$\sigma = \frac{q}{4\pi R^2}$$

Case I — point outside the shell ($r > R$): draw a spherical Gaussian surface of radius r , concentric with the shell. At every point of this surface the field and the area vector point in the same direction, so by Gauss's law:

$$\oint \vec{E} \cdot d\vec{S} = \frac{q}{\epsilon_0} \Rightarrow E(4\pi r^2) = \frac{4\pi R^2 \sigma}{\epsilon_0}$$

FIELD OUTSIDE A CHARGED SPHERICAL SHELL

$$E = \frac{\sigma R^2}{\epsilon_0 r^2}$$

Case II — point inside the shell ($r < R$): a Gaussian surface drawn inside the shell encloses no charge, so:

FIELD INSIDE A CHARGED SPHERICAL SHELL

$$q_{in} = 0 \Rightarrow \oint \vec{E} \cdot d\vec{S} = 0 \Rightarrow E = 0$$

Electric Field due to a Thin Infinite Sheet of Charge

Consider an infinite plane sheet carrying a uniform positive surface charge density σ . Take two points P and P' at equal distances on either side of the sheet, and draw a cylindrical Gaussian surface through them, with its axis perpendicular to the sheet, enclosing the whole sheet.

At the curved surface of this cylinder the field is perpendicular to the area vector, so there is no flux there; at the two flat end faces the field is parallel to the area vector:

$$d\phi_{\text{curved}} = E dS \cos 90^\circ = 0$$

$$d\phi_{\text{faces}} = E dS \cos 0^\circ + E dS \cos 0^\circ = 2E dS$$

Applying Gauss's law:

$$\oint \vec{E} \cdot d\vec{S} = \frac{q}{\epsilon_0} \Rightarrow 2E \oint dS = \frac{q}{\epsilon_0}$$

FIELD DUE TO AN INFINITE SHEET OF CHARGE

$$E = \frac{\sigma}{2\epsilon_0}$$

Electric Field due to a Thick Plane Sheet of Charge

Consider a thick plane sheet of charge of thickness a , and a point P at a distance r from its front face. The field due to the sheet at this point is:

$$E_1 = \frac{\sigma}{2\epsilon_0}$$

Electric Field Intensity Due to a Thick Sheet of Charge

The electric field due to a charged sheet does not depend on its thickness. So for a thick (conducting) charged sheet, the field at an external point P due to the back face still acts along the same direction as the field due to the front face — the two add up.

FIELD OUTSIDE A THICK CHARGED SHEET

$$E_{\text{thick sheet}} = E_1 + E_2 = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

Electric Field Intensity Due to Thin Sheets of Charge

Consider two thin, parallel, infinite sheets of charge, sheet I (surface charge density σ_1) and sheet II (surface charge density σ_2), facing each other. Three points are of interest: P (outside sheet I, to its left), the region between the two sheets (point Q), and R (outside sheet II, to its right).

Field due to each sheet alone, on either side of it:

$$E_1 = \frac{\sigma_1}{2\epsilon_0}, \quad E_2 = \frac{\sigma_2}{2\epsilon_0}$$

Taking the direction from sheet I to sheet II as positive, and adding the two fields with their correct directions at each point:

- At R (outside, beyond sheet II): both fields point away from the sheets, in the same direction — $E_R = E_1 + E_2 = \frac{\sigma_1 + \sigma_2}{2\epsilon_0}$
- At P (outside, beyond sheet I): both fields again point away from the sheets, but in the opposite direction to R — $E_P = -(E_1 + E_2) = -\frac{\sigma_1 + \sigma_2}{2\epsilon_0}$
- At Q (between the sheets): the two fields point towards each other, so they subtract — $E_Q = E_1 - E_2 = \frac{\sigma_1 - \sigma_2}{2\epsilon_0}$

Capacitor

A capacitor is a device which can store a large amount of charge and energy in a small space. The charge given to a conductor is directly proportional to its potential:

$$q \propto V \Rightarrow q = CV \Rightarrow C = \frac{q}{V}$$

where C is called the capacitance of the capacitor.

Principle of a Capacitor

- It is based on the fact that a capacitor can store a large amount of energy in a small space by placing an uncharged, earthed conductor near a charged conductor.
- It is based on the phenomenon of electrostatic induction.
- Charge on the plates of a capacitor is equal in magnitude and opposite in sign.
- Electric field intensity between the plates of the capacitor remains uniform, except near the edges of the plates.

- If a dielectric of dielectric constant K is filled between the plates, the field between the plates becomes $E_m = \frac{q}{K\epsilon_0 A}$.

Parallel Plate Capacitor

A parallel plate capacitor consists of two parallel, plane sheets of charge (plates), each of area A , separated by a distance d , with air/vacuum initially filling the gap between them.

Electric field intensity between the plates:

$$E_0 = \frac{\sigma}{\epsilon_0} = \frac{q}{A\epsilon_0}$$

With potential difference $V = E_0 d$, the capacitance of the parallel plate capacitor is:

CAPACITANCE OF A PARALLEL PLATE CAPACITOR (AIR/VACUUM)

$$C_0 = \frac{q}{V} = \frac{q}{E_0 d} = \frac{\epsilon_0 A}{d}$$

The capacitance of a parallel plate capacitor depends upon:

- area of the plates, A
- separation between the plates, d
- the medium between the plates

If a dielectric of dielectric constant K completely fills the space between the plates, the capacitance becomes K times its vacuum value: $C = KC_0$.

Parallel Plate Capacitor with a Conducting Slab

Consider a parallel plate capacitor (plate area A , separation d , charge $\pm q$) with air/vacuum between the plates. A conducting slab of thickness t ($t < d$) is now inserted between the plates without touching either plate.

Since the field and charge inside a conductor are always zero, the field E_0 due to the plates exists only in the remaining $(d - t)$ gap of air; it vanishes across the thickness of the slab. The potential difference between the plates is therefore:

$$V = E_0(d - t) = \frac{q}{A\epsilon_0}(d - t)$$

So the new capacitance is:

CAPACITANCE WITH A CONDUCTING SLAB INSERTED

$$C = \frac{q}{V} = \frac{A\epsilon_0}{d - t} = \frac{A\epsilon_0}{d(1 - \frac{t}{d})} = \frac{C_0}{1 - \frac{t}{d}}$$

Dielectrics

Dielectrics are a special type of insulator which transmit electric effects without conduction.

Polar dielectric — a dielectric in which the centres of the positive and negative charge of a molecule are naturally separated, so it has a non-zero dipole moment even with no external field.

Non-polar dielectric — a dielectric in which the centres of the positive and negative charge of a molecule coincide, so it has zero dipole moment with no external field.

Note: when a non-polar dielectric is placed in an external electric field, its charges begin to separate until a stage is reached where the non-polar dielectric effectively becomes a polar one — it acquires a temporary, induced dipole moment.

Note: this polarisation of charge sets up an electric field inside the dielectric that opposes the direction of the applied field. So the net field inside the dielectric is:

$$\vec{E} = \vec{E}_0 - \vec{E}_p$$

where $\vec{E}_0$ is the applied (external) field and $\vec{E}_p$ is the opposing field set up by the induced polarisation.

Capacitance of a Parallel Plate Capacitor with a Dielectric Slab

Consider a parallel plate capacitor of plate area A and charge $\pm q$, with field $E_0 = \sigma/\epsilon_0 = q/(A\epsilon_0)$ and capacitance $C_0 = \epsilon_0 A/d$ between the plates (air/vacuum). A dielectric slab of dielectric constant K and thickness t is now placed between the plates.

Polarisation of the dielectric sets up an induced field opposing E_0 , so the resultant (net) field inside the dielectric slab drops to E_0/K , while the field in the remaining air gap (of thickness $d - t$) stays E_0 . The potential difference across the plates is then the sum of the drop across the air gap and the drop across the slab:

$$V = E_0(d - t) + \frac{E_0}{K}t = E_0 \left[d - t + \frac{t}{K} \right]$$

giving the capacitance:

CAPACITANCE WITH A DIELECTRIC SLAB INSERTED

$$C = \frac{q}{V} = \frac{A\epsilon_0}{d - t + \frac{t}{K}} = \frac{A\epsilon_0}{d \left[1 - \frac{t}{d} \left(1 - \frac{1}{K} \right) \right]} = \frac{C_0}{1 - \frac{t}{d} \left(1 - \frac{1}{K} \right)}$$

Series Combination of Capacitors

- Capacitors are said to be in series if the second plate of the first capacitor is connected to the first plate of the second capacitor, the second plate of the second is connected to the first plate of the third, and so on.

- During this, the charge on each capacitor remains the same, but the potential difference across each is different: $V_1, V_2, \dots, V_n$.
- The equivalent capacitance of a series combination is always less than the smallest individual capacitance (charge same, V different, equivalent capacitance is minimum).

Let capacitors of capacitance $C_1, C_2, \dots, C_n$ be joined end-to-end (in series) across a source of potential difference V . Each carries the same charge q , and the source potential splits across them:

$$V = V_1 + V_2 + V_3 + \dots + V_n, \quad V_1 = \frac{q}{C_1}, V_2 = \frac{q}{C_2}, \dots, V_n = \frac{q}{C_n}$$

so that:

EQUIVALENT CAPACITANCE IN SERIES

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_n}$$

where C_S is the equivalent capacitance of the series combination.

Parallel Combination of Capacitors

Capacitors are said to be in parallel if one plate of each capacitor is joined at a common point, the second plate of each is joined at another common point, and these two points are connected to a source. During this, the potential difference across each capacitor remains the same, but the charge on each is different.

- In a parallel combination, the equivalent capacitance is maximum (greater than the largest individual capacitance), charge is different across each capacitor, and V is the same across each.

Let capacitors $C_1, C_2, \dots, C_n$ be connected in parallel across a source of potential difference V . The total charge drawn from the source splits between them:

$$q = q_1 + q_2 + q_3 + \dots + q_n, \quad q_1 = C_1 V, q_2 = C_2 V, \dots, q_n = C_n V$$

Since $q = C_P V$ for the combination:

EQUIVALENT CAPACITANCE IN PARALLEL

$$C_P = C_1 + C_2 + C_3 + \dots + C_n$$

where C_P is the equivalent capacitance of the parallel combination.

Worked problem

Find the charge and potential difference across each capacitor in the network shown, connected to a 300 V source. The network has: $C_1 = 100$ pF directly across the two main nodes; $C_2 = 200$ pF in series with $C_3 = 200$ pF, also connected across the same two nodes (so this series pair forms a second path alongside C_1); and $C_4 = 100$ pF in series with this whole combination, across the 300 V supply.

First, reduce the network step by step.

$$\frac{1}{C_{23}} = \frac{1}{C_2} + \frac{1}{C_3} = \frac{1}{200} + \frac{1}{200} \Rightarrow C_{23} = 100 \text{ pF}$$

C_1 and the C_2 – C_3 series pair are both connected across the same two nodes, so they are in parallel:

$$C_x = C_1 + C_{23} = 100 + 100 = 200 \text{ pF}$$

This combination C_x is in series with C_4 across the 300 V source:

$$\frac{1}{C_{eq}} = \frac{1}{C_x} + \frac{1}{C_4} = \frac{1}{200} + \frac{1}{100} = \frac{3}{200} \Rightarrow C_{eq} = \frac{200}{3} \text{ pF}$$

Total charge drawn from the source:

$$Q = C_{eq}V = \frac{200}{3} \times 300 = 2 \times 10^4 \text{ pF} \cdot \text{V} = 2 \times 10^{-8} \text{ C}$$

This entire charge Q flows through C_4 (the single series element), and also appears across the block C_x as a whole:

$$V_4 = \frac{Q}{C_4} = \frac{2 \times 10^{-8}}{100 \times 10^{-12}} = 200 \text{ V}, \quad V_x = \frac{Q}{C_x} = \frac{2 \times 10^{-8}}{200 \times 10^{-12}} = 100 \text{ V}$$

(Check: $V_4 + V_x = 200 + 100 = 300 \text{ V}$, the full supply.) Inside the block, C_1 (100 pF) and the C_2 – C_3 series pair (100 pF) are equal, so the charge Q splits equally between them:

$$q_1 = q_{23} = \frac{Q}{2} = 1 \times 10^{-8} \text{ C} \Rightarrow V_1 = \frac{q_1}{C_1} = \frac{10^{-8}}{100 \times 10^{-12}} = 100 \text{ V}$$

C_2 and C_3 , being in series, carry this same charge:

$$q_2 = q_3 = 1 \times 10^{-8} \text{ C} \Rightarrow V_2 = \frac{q_2}{C_2} = 50 \text{ V}, \quad V_3 = \frac{q_3}{C_3} = 50 \text{ V}$$

Answer: C_1 : $q = 1 \times 10^{-8} \text{ C}$, $V = 100 \text{ V}$ | C_2 : $q = 1 \times 10^{-8} \text{ C}$, $V = 50 \text{ V}$ | C_3 : $q = 1 \times 10^{-8} \text{ C}$, $V = 50 \text{ V}$ | C_4 : $q = 2 \times 10^{-8} \text{ C}$, $V = 200 \text{ V}$.

Worked problem

An electrician has a number of capacitors, each of $1 \mu\text{F}$, and needs an equivalent capacitance of $2 \mu\text{F}$. Give an arrangement to obtain this if the supply voltage is 1 kV and one capacitor can withstand a maximum potential of 400 V .

Let n identical $1 \mu\text{F}$ capacitors be joined in series, so the 1000 V supply divides equally across them, with each capacitor seeing at most 400 V :

$$400 \times n \geq 1000 \Rightarrow n \geq \frac{1000}{400} = 2.5$$

Since n must be a whole number, $n = 3$ capacitors are connected in series per string (each then sees about 333 V, safely under the 400 V limit). The equivalent capacitance of one such series string is:

$$\frac{1}{C_s} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1} = 3 \Rightarrow C_s = \frac{1}{3} \mu\text{F}$$

(The notebook continues this on the next page, combining several such 3-capacitor series strings in parallel to build up the full $2 \mu\text{F}$.)

Answer: 3 capacitors of $1 \mu\text{F}$ each, connected in series, form one string of equivalent capacitance $1/3 \mu\text{F}$ able to withstand the 1 kV supply safely.

Continuing a combination problem from the previous page: identical $1 \mu\text{F}$ capacitors are grouped into rows of three in series, and several such rows are placed in parallel, to build up a required equivalent capacitance of $2 \mu\text{F}$.

$$C_s = \frac{1}{3} \mu\text{F} \quad (\text{three } 1 \mu\text{F} \text{ capacitors in series, per row})$$

Suppose m such rows are connected in parallel. For the combination to equal $2 \mu\text{F}$:

$$\frac{1}{3} \times m = 2 \Rightarrow m = 6$$

So 6 rows of 3 capacitors each are needed:

$$\text{Number of capacitors} = 6 \times 3 \quad (m \times n) = 18$$

Wheatstone Bridge of Capacitors

Five capacitors can be wired into a bridge, exactly like a resistor Wheatstone bridge. C_1 and C_2 form the top arms (A to a midpoint C, then C to B), C_3 and C_4 form the bottom arms (A to a midpoint D, then D to B), and C_5 bridges the two midpoints C and D.

The bridge is balanced when:

BALANCE CONDITION

$$\frac{C_1}{C_2} = \frac{C_3}{C_4}$$

- No charge sits on the bridging capacitor C_5 , so it can simply be removed from the circuit.
- The two midpoints, C and D, are then at the same potential: $V_C - V_D = 0$.

Worked problem

The network below is a balanced bridge. Find the value of x .

From the input terminal A, two capacitors 2 and 4 in parallel (giving 6) lead to a top midpoint; from A, two $2\ \mu\text{F}$ capacitors in series (giving an equivalent of 1) lead to a bottom midpoint. An uncharged bridging capacitor connects the two midpoints. From the top midpoint, capacitors 3 and x in parallel (giving $3 + x$) lead to the output terminal B; from the bottom midpoint, a single capacitor 4 leads to B.

Applying the balance condition:

$$\frac{6}{3 + x} = \frac{1}{4}$$

$$24 = 3 + x$$

Answer: $x = 21$

Capacitance of Infinite Ladder Networks

Worked problem

Find the equivalent capacitance of the network shown: a first branch is a single capacitor C ; a second branch is two capacitors C in series; a third branch is four capacitors C in series; and so on, doubling each time (8, 16, ..., without end), with every branch connected in parallel between the same two rails.

Each branch of 2^n series capacitors C is equivalent to $C/2^n$, so the branches are C , $C/2$, $C/4$, $C/8$, ..., all in parallel:

$$C_E = C + \frac{C}{2} + \frac{C}{4} + \frac{C}{8} + \dots$$

This is an infinite geometric series with first term C and common ratio $\frac{1}{2}$ (sum $S_\infty = a/(1 - r)$):

$$C_E = \frac{C}{1 - \frac{1}{2}}$$

Answer: $C_E = 2C$

Worked problem

A 6 V source is connected to a ladder network that repeats forever: a $2\ \mu\text{F}$ capacitor in series along the top rail, then a $1\ \mu\text{F}$ capacitor as a shunt down to the bottom rail, then another $2\ \mu\text{F}$ in series, another $1\ \mu\text{F}$ shunt, and so on without end. Find the capacitance of the whole network.

Let the capacitance of the infinite network, as seen by the source, be x . Because the ladder repeats forever, removing the very first $2\ \mu\text{F}$ – $1\ \mu\text{F}$ section leaves a network that is still identical to the whole thing — still x . So the whole network is just the first $2\ \mu\text{F}$ in series with (the first $1\ \mu\text{F}$ in parallel with another copy of the whole network, x):

$$\frac{1}{x} = \frac{1}{2} + \frac{1}{1+x}$$

Cross-multiplying:

$$2(1+x) = x(x+3) \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0$$

giving $x = 1$ or $x = -2$. Capacitance cannot be negative, so:

Answer: $x = 1 \mu\text{F}$

Potential at an Interior Point of a Series Combination

Worked problem

Terminal A is held at 1000 V. A $3 \mu\text{F}$ capacitor connects A to a point P, and a $2 \mu\text{F}$ capacitor connects P to terminal B, which is grounded (0 V). Find the potential of P.

The two capacitors are in series, so the same charge sits on both:

$$\frac{1}{C} = \frac{1}{3} + \frac{1}{2} \Rightarrow C = \frac{6}{5} \mu\text{F}$$

$$q = CV = \frac{6}{5} \times 1000 = 1200 \mu\text{C}$$

The same charge sits on the $3 \mu\text{F}$ capacitor (between A and P) and on the $2 \mu\text{F}$ capacitor (between P and B):

$$q = C_1(V_A - V_P) \Rightarrow 1200 = 3(1000 - V_P) \Rightarrow V_P = 600 \text{ V}$$

$$q = C_2(V_P - V_B) \Rightarrow 1200 = 2(V_P - 0) \Rightarrow V_P = 600 \text{ V}$$

Answer: $V_P = 600 \text{ V}$, consistent both ways.

Energy Stored in a Capacitor

The work done by a source charging a capacitor is stored as electrostatic potential energy. At some stage the capacitor already carries charge q , at potential $V = q/C$. Bringing a further small charge dq onto it costs a small amount of work:

$$dW = V dq = \frac{q}{C} dq$$

Adding up this work from zero charge up to the final charge Q gives the total work done, which is the energy stored:

$$W = \frac{1}{C} \int_0^Q q \, dq = \frac{1}{C} \left(\frac{Q^2}{2} \right) = \frac{Q^2}{2C}$$

ENERGY STORED IN A CAPACITOR

$$U = \frac{Q^2}{2C}$$

Using $Q = CV$, this can equally be written as:

$$U = \frac{1}{2}CV^2$$

When a battery of EMF V charges an initially uncharged capacitor through a connecting wire, the total work done by the battery is $W = QV = CV^2$, twice the energy that ends up stored. The rest is lost as heat in the (unavoidable) resistance of the connecting wire, however small that resistance is:

WORK DONE BY THE SOURCE

$$W = \frac{1}{2}CV^2 + H$$

$$\text{Heat loss } H = \frac{1}{2}CV^2$$

So $W = \frac{1}{2}CV^2 + \frac{1}{2}CV^2 = CV^2 = qV$: while charging a capacitor, exactly 50% of the source's energy is stored in the capacitor, and the other 50% is always lost as heat in the connecting wire.

Energy Stored in a Series Combination of Capacitors

For capacitors $C_1, C_2, C_3, \dots$ in series across a source V , the same charge Q_s sits on every capacitor, and:

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots$$

Total energy stored, written using the series capacitance:

$$E = \frac{Q_s^2}{2C_s} = \frac{Q_s^2}{2} \left[\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots \right]$$

$$E = \frac{Q_s^2}{2C_1} + \frac{Q_s^2}{2C_2} + \frac{Q_s^2}{2C_3} + \dots$$

and since Q_s is the charge on every individual capacitor too, each term is just that capacitor's own stored energy:

$$E = E_1 + E_2 + E_3 + \dots$$

Energy Stored in a Parallel Combination of Capacitors

For capacitors in parallel across the same source V , $C_p = C_1 + C_2 + \dots$, and:

$$E = \frac{1}{2}C_p V^2 = \frac{1}{2}V^2(C_1 + C_2 + \dots) = \frac{1}{2}V^2 C_1 + \frac{1}{2}V^2 C_2 + \dots$$

$$E = E_1 + E_2 + \dots$$

In both series and parallel combinations, then, the total energy stored is simply the sum of the energies that would be stored in each capacitor on its own, for the same capacitance values and the same source.

Energy Density of a Parallel-Plate Capacitor

Energy stored per unit volume of a parallel-plate capacitor is called its **energy density**:

$$\text{Energy Density} = \frac{\text{Energy Stored}}{\text{Volume of capacitor}} = \frac{\frac{1}{2}CV^2}{A \cdot d}$$

Substituting $C = \epsilon_0 A/d$ and $V = Ed$:

ENERGY DENSITY

$$u = \frac{1}{2}\epsilon_0 E^2$$

Common Potential

Consider two conductors A and B, holding charges q_1 and q_2 at potentials V_1 and V_2 (say $V_1 > V_2$), with capacitances C_1 and C_2 . The total charge, before anything is connected, is:

$$q' = C_1 V_1 + C_2 V_2$$

Now join A and B with a connecting wire. Charge flows from the higher potential to the lower one, from A to B, until a stage is reached where both conductors are at the same potential, the **common potential** V . At that point each conductor's charge is just its own capacitance times V :

$$q' = C_1 V + C_2 V = (C_1 + C_2)V$$

By conservation of charge, this must equal the original total charge:

$$(C_1 + C_2)V = C_1V_1 + C_2V_2$$

COMMON POTENTIAL

$$V = \frac{C_1V_1 + C_2V_2}{C_1 + C_2}$$

Loss of Energy During Charge Sharing

Total energy before sharing:

$$E_i = \frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2$$

Total energy after sharing, using the common potential:

$$E_f = \frac{1}{2}(C_1 + C_2)V^2 = \frac{(C_1V_1 + C_2V_2)^2}{2(C_1 + C_2)}$$

The loss in energy, $\Delta E = E_i - E_f$, simplifies to:

LOSS OF ENERGY ON SHARING

$$\Delta E = \frac{C_1C_2(V_1 - V_2)^2}{2(C_1 + C_2)}$$

This energy is never negative, and it is zero only when $V_1 = V_2$ to begin with: it is dissipated as heat and a brief spark in the connecting wire.

Worked problem

Find the charge across each capacitor: a 1200 V source connects through a 4 μF capacitor to a node that holds a 2 μF capacitor and a 3 μF capacitor in parallel, the far side of which is grounded.
(The notebook's worked solution for this one continues past the pages captured in this excerpt.)

Solving Capacitors Filled With Two Different Dielectric Media

When a parallel-plate capacitor is filled with two different media (or split into separate regions), it reduces to a series/parallel combination depending on how it is split:

- **Area division** — splitting the plates side by side (same separation, different areas) is equivalent to two capacitors in **parallel**.
- **Distance division** — splitting the gap into layers along the separation direction (same area, each layer a fraction of the separation) is equivalent to two capacitors in **series**.
- **Both together** — splitting by area and by distance at once gives a combination of series and parallel capacitors.

Worked problem

Four identical plates, each of area A , are arranged with separation d between every pair of consecutive plates, as a parallel-plate capacitor. Find the equivalent capacitance for each of the two connection schemes shown, (a) and (b).

In (a), the four plates are wired so that the stack behaves as two identical capacitors of capacitance $\epsilon_0 A/d$ in parallel; in (b), they are wired so that it behaves as three such capacitors in parallel.

$$(a) \ C = \frac{2\epsilon_0 A}{d}$$

$$(b) \ C = \frac{3\epsilon_0 A}{d}$$

Answer: $C_{(a)} = 2\epsilon_0 A/d$, $C_{(b)} = 3\epsilon_0 A/d$

Star–Delta Method for Capacitor Networks

Some capacitor bridges are not balanced and cannot be reduced by plain series/parallel steps. One way through is to redraw three capacitors that meet at a common point (a “star”, or Y) as an equivalent triangle of three capacitors directly joining their three outer nodes (a “delta”, or Δ), which removes the shared point from the circuit entirely.

Worked problem

Terminals A and B are joined by a $1\ \mu\text{F}$ capacitor in series with a $3\ \mu\text{F}$ capacitor (through a midpoint M on top), by a $3\ \mu\text{F}$ capacitor in series with a $1\ \mu\text{F}$ capacitor (through a midpoint N on the bottom), and by a bridging $4\ \mu\text{F}$ capacitor directly between M and N. Find the equivalent capacitance between A and B.

The three capacitors meeting at M — $1\ \mu\text{F}$ (to A), $3\ \mu\text{F}$ (to B), and $4\ \mu\text{F}$ (to N) — form a star. Converting to the equivalent delta directly joining A, B and N (for a star of C_1, C_2, C_3 , each delta side is $C_i C_j / (C_1 + C_2 + C_3)$):

$$C_{AB} = \frac{1 \times 3}{1 + 3 + 4} = \frac{3}{8}, \quad C_{AN} = \frac{1 \times 4}{1 + 3 + 4} = \frac{4}{8} = \frac{1}{2}, \quad C_{BN} = \frac{3 \times 4}{1 + 3 + 4} = \frac{12}{8} = \frac{3}{2}$$

M has now been eliminated. The new C_{AN} and C_{BN} sit in parallel with the original bottom-arm capacitors between the same pairs of nodes ($3\ \mu\text{F}$ between A and N, $1\ \mu\text{F}$ between N and B):

$$C_{AN}^{\text{total}} = \frac{1}{2} + 3 = \frac{7}{2}, \quad C_{BN}^{\text{total}} = \frac{3}{2} + 1 = \frac{5}{2}$$

These two are in series, forming a path from A to B through N, which then sits in parallel with the direct $C_{AB} = 3/8$:

$$\frac{1}{C_{ANB}} = \frac{2}{7} + \frac{2}{5} = \frac{24}{35} \Rightarrow C_{ANB} = \frac{35}{24}$$

$$C_{eq} = \frac{3}{8} + \frac{35}{24} = \frac{9}{24} + \frac{35}{24} = \frac{44}{24}$$

Answer: $C_{eq} = \frac{11}{6} \mu\text{F}$

Worked problem

Equal charges q are placed at each vertex A, B, C of an equilateral triangle of side l , and a charge Q is placed at the centroid O. Find the net force on Q .

Let D be the midpoint of BC, so AD is a median of the triangle.

$$AD = \sqrt{l^2 - \frac{l^2}{4}} = \frac{\sqrt{3}}{2} l$$

Since O is the centroid, it divides each median in the ratio 2:1 from the vertex:

$$AO = \frac{2}{3} AD = \frac{l}{\sqrt{3}}$$

By symmetry, $AO = OB = OC$, so the force of repulsion between each vertex charge and Q is the same size:

FORCE BETWEEN Q AND EACH VERTEX CHARGE

$$F_{OA} = F_{OB} = F_{OC} = \frac{3kQq}{l^2}$$

F_{OB} and F_{OC} act at 120° to each other (the angle $\angle BOC$). Their resultant is:

$$F_{BC} = \sqrt{F_{OB}^2 + F_{OC}^2 + 2F_{OB}F_{OC} \cos 120^\circ} = F_{OB}$$

This resultant is equal in magnitude to F_{OA} and points directly opposite to it, so the net force on Q is zero.

$$F_{net} = F_{BC} - F_{OA} = 0$$

Answer: The net force on Q is zero, by the symmetry of the configuration.

Worked problem

Charges $+q$, $+q$ are placed at the two base vertices of an equilateral triangle of side l , and a charge $-q$ is placed at the apex. Find the force on each charge.

Every pair of charges is separated by the same distance l , so every pairwise force has the same magnitude:

$$F = \frac{kq^2}{l^2}$$

Force on the apex ($-q$) charge: it is attracted toward each base charge along the two sides of the triangle, and these two attractive forces meet at the triangle's apex angle of 60° .

FORCE ON THE -Q CHARGE

$$F_{-q} = \sqrt{F^2 + F^2 + 2F^2 \cos 60^\circ} = \sqrt{3} \frac{kq^2}{l^2}$$

It is directed straight from the apex toward the midpoint of the base.

Force on each base (+q) charge: it is attracted toward the apex along one side, and repelled away from the other base charge along the base. These two forces act at 120° to each other.

FORCE ON EACH +Q CHARGE

$$F_{+q} = \sqrt{F^2 + F^2 + 2F^2 \cos 120^\circ} = \frac{kq^2}{l^2}$$

Answer: The apex charge feels a force $\sqrt{3} kq^2/l^2$ pulling it toward the base; each base charge feels a smaller force kq^2/l^2 pushing it outward, away from the triangle.

Current Electricity

The flow of electric charges in a conductor constitutes an electric current in the conductor. The charged particles that carry this current in solids, liquids, and gases are known as current carriers.

Electric current is the rate of flow of charge through the cross-section of a conductor:

ELECTRIC CURRENT

$$I = \frac{Q}{t} = \frac{ne}{t}$$

where n is the number of electrons crossing the cross-section of the conductor in time t .

Note: although current has a magnitude and a direction, it is a scalar quantity — it does not obey the laws of vector addition. Net current is found by simple algebraic addition.

Types of Current

- **Steady direct current:** magnitude and direction do not change with time.
- **Varying (variable) direct current:** magnitude changes with time, but polarity stays the same.
- **Alternating current:** magnitude changes with time and polarity reverses periodically, i.e. after a regular interval of time.

Current that never reverses polarity is called direct current (DC) — this covers both steady and varying current of the same polarity. Current whose polarity reverses periodically is called alternating current (AC).

Drift Velocity

In the absence of an electric field, the average thermal velocity of the free electrons in a conductor is zero. When an electric field is applied across the conductor, all the electrons drift against the field.

The velocity with which the electrons of a conductor drift from lower to higher potential (i.e. against the electric field) is called the drift velocity, or drift speed.

The drift velocity of an electron in a conductor, about 10^{-3} to 10^{-5} m/s, is extremely small — but its effect is felt almost instantly because the electric field itself is established across the conductor at close to the speed of light, 3×10^8 m/s.

Expression for Drift Velocity

Consider a conductor of length l and cross-sectional area A . In the absence of an electric field, let $\vec{u}_1, \vec{u}_2, \vec{u}_3, \dots, \vec{u}_n$ be the random thermal velocities of the free electrons. Being random in direction, these average to zero:

$$\frac{\vec{u}_1 + \vec{u}_2 + \dots + \vec{u}_n}{n} = 0$$

In the presence of an electric field, every electron picks up an additional velocity against the field. For an electron of charge $-e$:

$$\vec{F} = m\vec{a} \Rightarrow -e\vec{E} = m\vec{a} \Rightarrow \vec{a} = -\frac{e\vec{E}}{m}$$

The negative sign shows that the electrons accelerate opposite to the direction of the electric field.

If T_i is the time since the i -th electron's last collision, its velocity becomes $\vec{v}_i = \vec{u}_i + \vec{a} T_i$. Averaging over all n electrons gives the drift velocity:

$$\vec{V}_d = \frac{\vec{v}_1 + \vec{v}_2 + \dots + \vec{v}_n}{n} = \underbrace{\frac{\vec{u}_1 + \dots + \vec{u}_n}{n}}_{=0} + \vec{a} \left(\frac{T_1 + T_2 + \dots + T_n}{n} \right) = \vec{a} \tau$$

where $\tau = \frac{T_1 + T_2 + \dots + T_n}{n}$ is the average time between collisions — the relaxation time. Substituting $\vec{a}$:

DRIFT VELOCITY

$$\vec{V}_d = -\frac{e\vec{E}\tau}{m}$$

Relation between Current and Drift Velocity

Consider a conductor of length l , cross-sectional area A , and volume $V = Al$. If n is the number of free electrons per unit volume, the total number of free electrons in the conductor is nAl , so the total charge they carry is $q = neAl$. All these electrons cross the conductor's cross-section in time t , so the current is:

$$I = \frac{q}{t} = \frac{neAl}{t} = neAV_d$$

CURRENT IN TERMS OF DRIFT VELOCITY

$$I = neAV_d$$

Ohm's Law

The current flowing in a conductor is directly proportional to the potential difference developed across it, provided the physical conditions (such as temperature) remain constant and there is no mechanical strain in the conductor.

$$I \propto V \Rightarrow V \propto I \Rightarrow V = IR$$

where R is the electrical resistance of the conductor.

Ohmic Devices

For ohmic devices, the V-I graph is a straight line through the origin, so the ratio V/I is constant. This constant ratio is also called the static resistance.

Non-Ohmic Devices

For non-ohmic devices, the V-I graph is not a straight line — it may flatten out (saturate) or curve sharply upward, depending on the device. The slope of a non-ohmic V-I curve at a point gives a variable resistance, also called the dynamic resistance.

Derivation of Ohm's Law from Drift Velocity

Consider a conductor of length l and cross-sectional area A , connected to a source of emf. A potential difference develops across the conductor, and all its free electrons move from the negative to the positive terminal with drift velocity V_d . Starting from $I = neAV_d$ and substituting $V_d = eE\tau/m$:

$$I = neA \left(\frac{eE\tau}{m} \right) = \frac{ne^2 AE\tau}{m}$$

Using $E = V/l$:

$$I = \frac{ne^2 A\tau}{m} \cdot \frac{V}{l} = \frac{ne^2 AV\tau}{ml}$$

Rearranging for V:

$$V = \left(\frac{ml}{ne^2 A\tau} \right) I = RI$$

RESISTANCE, FROM THE DRIFT-VELOCITY MODEL

$$R = \frac{ml}{ne^2 A\tau}$$

So R is directly proportional to length and inversely proportional to area:

$$R \propto l, \quad R \propto \frac{1}{A}, \quad \rho = \frac{m}{ne^2 \tau}$$

ρ is constant for a given material and does not depend on the conductor's dimensions (length or area of cross-section). It is a property of the material itself, called the resistivity or specific resistance.

Effect of Stretching a Wire on Resistance

(1) If a wire of resistance R and length l is stretched so that its length becomes n times the original (its volume stays the same), the new resistance is:

WIRE STRETCHED TO n TIMES ITS LENGTH

$$R' = n^2 R$$

Proof: with $R_1 = \rho l_1 / A_1$ and $R_2 = \rho l_2 / A_2$, and $l_2 = n l_1$. Since the volume is unchanged, $A_1 l_1 = A_2 l_2$, so $A_1 / A_2 = l_2 / l_1 = n$. Then:

$$\frac{R_1}{R_2} = \frac{l_1}{l_2} \times \frac{A_2}{A_1} = \frac{1}{n} \times \frac{1}{n} = \frac{1}{n^2} \Rightarrow R_2 = n^2 R_1$$

(2) If instead a wire of resistance R is stretched so that its radius (diameter/thickness) becomes $1/n$ times the original — i.e. the radius scales down and the length scales up to keep the volume constant — the new resistance is:

WIRE STRETCHED SO ITS RADIUS SCALES BY A FACTOR n (VOLUME CONSTANT)

$$R' = \frac{R}{n^4}$$

Proof: with $R_1 = \rho l_1 / A_1$, $R_2 = \rho l_2 / A_2$, and $r_2 = n r_1$. Volume is conserved, $l_1 A_1 = l_2 A_2$, so $l_1 / l_2 = A_2 / A_1 = r_2^2 / r_1^2 = n^2$. Then:

$$\frac{R_1}{R_2} = \frac{l_1}{l_2} \times \frac{A_2}{A_1} = n^2 \times n^2 = n^4 \Rightarrow R_2 = \frac{R_1}{n^4}$$

Electrical Conductivity

The reciprocal of the resistivity of a conductor is called its electrical conductivity.

$$\sigma = \frac{1}{\rho}$$

Conductance

The reciprocal of electrical resistance is called conductance.

$$C = \frac{1}{R}$$

Current Density

The current flowing per unit area of cross-section of a conductor is called the current density.

$$J = \frac{I}{A}$$

Mobility of Charge Carrier

The drift velocity per unit electric field is called the mobility of a charge carrier.

$$\mu = \frac{V_d}{E}$$

Relation between Electric Field, Conductivity and Current Density

Starting from $I = neAV_d$:

$$\frac{I}{A} = neV_d \Rightarrow J = neV_d$$

Substituting $V_d = eE\tau/m$:

$$J = \frac{ne^2 E \tau}{m}$$

MICROSCOPIC (OHM'S LAW) FORM

$$J = \sigma E, \quad \sigma = \frac{ne^2 \tau}{m}$$

Effect of Temperature on Resistance

From $R = ml/(ne^2 A \tau)$, resistance depends on the relaxation time τ : R increases with temperature T, and R is inversely proportional to τ .

$$R \propto T, \quad R \propto \frac{1}{\tau}$$

When the temperature of a conductor increases, the relaxation time decreases (more frequent collisions) and so the resistance increases.

RESISTANCE AT TEMPERATURE T

$$R_t = R_0(1 + \alpha \Delta T)$$

$$\alpha = \frac{R_t - R_0}{R_0 \Delta T}$$

where:

- α = temperature coefficient of resistance
- R_0 = resistance at 0°C
- R_t = resistance at the unknown temperature
- **Metals:** α is positive ($T \uparrow$ so $R \uparrow$).
- **Semiconductors:** α is negative ($T \uparrow$ so $R \downarrow$).
- **Alloys:** α is positive, but very small.

Electrical Energy and Power

The work done by a source to maintain a current in an electric circuit is known as electrical energy. For a resistor R carrying current I with potential difference V across it, the charge transferred in time t is $q = It$, so the work done is:

$$q = It$$

ELECTRICAL ENERGY

$$W = Vq = VIt$$

This work done is equal to the electrical energy consumed in the circuit.

The electrical energy consumed by a device in time t is $E = VIt = I^2Rt = \frac{V^2t}{R}$. The S.I. unit of electrical energy is the joule (J).

Electric Power

Electric power is the rate of doing electrical work.

$$P = \frac{E}{t} = VI = I^2R = \frac{V^2}{R}$$

The S.I. unit of electrical power is the watt (W).

Cell

A cell is a device which provides the necessary potential difference to an electric circuit to maintain a continuous flow of current in it.

EMF (Electromotive Force)

The EMF of a cell may be defined as the potential difference between the terminals of the cell when no current is drawn from the cell.

Terminal Potential Difference

The terminal potential difference of a cell is defined as the potential difference between its terminals in a closed circuit (i.e. when current is drawn from the cell).

Internal Resistance

The internal resistance of a cell is defined as the opposition offered by the electrolyte and electrodes of a cell to the flow of current through it. It is denoted by r and mainly depends on the nature of the electrolyte and electrodes of the cell. A cell of EMF E and internal resistance r is represented by the usual cell symbol drawn in series with a resistor r .

- Direction of conventional current inside the cell is from the negative to the positive electrode, whereas outside the cell it is from the positive to the negative electrode.
- Internal resistance of a cell increases with the passage of time and usage of the cell.
- Internal resistance of a cell increases with a decrease in temperature.

Relation Between Terminal Potential Difference and EMF

A cell of EMF E and internal resistance r is connected to an external resistance R , forming a closed circuit. The potential difference across the terminals of the cell is called the terminal potential difference.

$$R_{eq} = r + R$$

$$i = \frac{E}{r + R}$$

Since $iR = V$, multiplying through and rearranging:

$$ir + iR = E \Rightarrow ir + V = E$$

TERMINAL POTENTIAL DIFFERENCE

$$V = E - ir$$

- $V < E$: current is being drawn from the cell, circuit is closed.
- $V = E \Rightarrow i = 0$: circuit is open.
- $V > E$: this happens during the charging process. During charging, the direction of current through the cell is reversed, so $V = E - (-i)R = E + iR$.

$$V = E + iR \quad (V > E, \text{ charging})$$

Maximum Power Transfer Theorem

If the internal resistance of a cell is equal to the external resistance of the circuit, the cell delivers maximum power to the circuit. This is called the maximum power transfer theorem.

$$i = \frac{E}{r + R}$$

When $r = R$:

$$i = \frac{E}{2R}$$

$$P_{max} = i^2 R = \frac{E^2}{4R}$$

Series Combination of Cells

n cells, each of EMF E and internal resistance r , are connected in series with an external resistance R .

- Equivalent EMF = nE
- Internal resistance = nr
- Total resistance = $nr + R$

TOTAL CURRENT

$$i = \frac{nE}{nr + R}$$

- If $nr \gg R$ (i.e. $R \approx 0$): $i = \frac{nE}{nr} = \frac{E}{r}$
- If $nr \ll R$ (i.e. $nr \approx 0$): $i = \frac{nE}{R}$

Parallel Combination of Cells

Cells are said to be in parallel if the positive terminal of each cell is connected to one common point and the negative terminal of each cell is connected to another common point. n such cells, each of EMF E and internal resistance r , are connected through an external resistance R .

- Total resistance = $\frac{r}{n} + R$

$$R_{eq} = \frac{r + nR}{n}$$

TOTAL CURRENT

$$i = \frac{E}{(r + nR)/n} = \frac{nE}{r + nR}$$

Kirchhoff's Junction Law

The algebraic sum of currents meeting at a junction is zero.

$$\sum I = 0$$

- Current coming toward a junction is equal to the current going away from the junction.
- Current coming toward the junction is taken as positive, and current going away from the junction is taken as negative.

For example, at a junction where currents i_1, i_2, i_3 flow in and currents i_4, i_5 flow out:

$$i_1 + i_2 + i_3 - i_4 - i_5 = 0$$

Kirchhoff's Voltage Law

In any closed loop, the algebraic sum of potential differences is equal to zero.

$$\sum IR - \sum E = 0$$

Or, in any closed loop, the algebraic sum of the products of resistance and current is equal to the algebraic sum of the EMFs of that loop.

$$\sum IR = \sum E$$

Sign Convention

Resistor (current flows from A to B through resistance R):

- Travelling A to B (along the current): $IR = +ve$
- Travelling B to A (opposite to the current): $IR = -ve$

Cell (travelling from the negative terminal A to the positive terminal B):

- Travelling A to B (negative to positive terminal): $E = +ve$
- Travelling B to A (positive to negative terminal): $E = -ve$

Worked problem

Using Kirchhoff's laws, determine the currents I_1 , I_2 and I_3 for the network shown below.

Three branches connect a node M to a node C: branch 1 carries current I_1 through a 24 V cell and a $2\ \Omega$ resistor; branch 2 carries current I_2 through a 27 V cell and a $6\ \Omega$ resistor; branch 3 carries current I_3 through a $4\ \Omega$ resistor. By the junction law at M, $I_1 = I_2 + I_3$.

Applying KVL to the loop through branches 1 and 2:

(1)

$$27 + 6I_2 + 2I_1 - 24 = 0 \Rightarrow 2I_1 + 6I_2 = -3$$

Applying KVL to the loop through branches 2 and 3:

(2)

$$27 + 6I_2 - 4I_3 = 0 \Rightarrow 6I_2 - 4I_3 = -27$$

Substituting $I_1 = I_2 + I_3$ into (1):

$$2(I_2 + I_3) + 6I_2 = -3 \Rightarrow 8I_2 + 2I_3 = -3$$

Multiplying by 2 and adding to (2): $16I_2 + 4I_3 = -6$, and $6I_2 - 4I_3 = -27$, giving $22I_2 = -33$.

Answer: $I_2 = -1.5 \text{ A}$, $I_3 = 4.5 \text{ A}$, and $I_1 = I_2 + I_3 = 3 \text{ A}$. The negative sign on I_2 means it actually flows opposite to the direction assumed in the figure.

Worked problem

Determine the current in each branch of the network shown in the figure.

Nodes A, B, D sit at the points of a diamond, each joined to A by a 4Ω branch (A–B carrying I_2 , A–D carrying $I_1 - I_2$); a diagonal branch straight across the middle (A–C) carries a 10 V cell in series with 1Ω and current I_1 . From B and D, 2Ω branches run in to the same midpoint C (carrying $I_2 + I_3$ and $I_1 - I_2 - I_3$ respectively), and plain connecting wires carry I_3 on from B and from D to a point E, where a 5 V cell closes the loop back to C.

Applying KVL to three independent loops of the network:

(1)

$$I_1 + 6I_2 + 2I_3 = 10$$

(2)

$$7I_1 - 6I_2 - 2I_3 = 10$$

(3)

$$-2I_1 + 4I_2 + 4I_3 = 5$$

Adding (1) and (2) eliminates I_2 and I_3 :

$$8I_1 = 20 \Rightarrow I_1 = 2.5 \text{ A}$$

Substituting $I_1 = 2.5$ into (1) and (3):

$$6I_2 + 2I_3 = 7.5 \quad \text{and} \quad 4I_2 + 4I_3 = 10$$

Solving these two equations simultaneously (multiply the first by 2 and subtract the second):

$$12I_2 + 4I_3 = 15, \quad 4I_2 + 4I_3 = 10 \Rightarrow 8I_2 = 5$$

Answer: $I_1 = 2.5 \text{ A}$, $I_2 = \frac{5}{8} \text{ A} = 0.625 \text{ A}$, and $I_3 = \frac{15}{8} \text{ A} = 1.875 \text{ A}$.

Wheatstone Bridge

A Wheatstone bridge has four resistance arms arranged between four nodes A, B, C, D: arm AB has resistance P (current I_1), arm BC has resistance Q (current $I_1 - I_g$), arm AD has resistance R (current $I - I_1$), and arm DC has resistance S (current $I - I_1 + I_g$). A galvanometer carrying current I_g is connected between B and D, and a cell (with a key) drives a total current I into A and out at C.

Applying KVL to loop ABDA:

(1)

$$GI_g + I_1P - (I - I_1)R = 0 \Rightarrow GI_g + I_1P = (I - I_1)R$$

Applying KVL to loop CDBC:

(2)

$$(I_1 - I_g)Q - (I - I_1 + I_g)S - I_gG = 0 \Rightarrow (I_1 - I_g)Q - I_gG = (I - I_1 + I_g)S$$

The resistance R is adjusted so that, on pressing the key, the galvanometer shows no deflection — the bridge is then said to be in the **balanced condition**, and no current flows through the galvanometer, i.e. $I_g = 0$.

Putting $I_g = 0$ in (1) and (2):

(3)

$$I_1P = (I - I_1)R$$

(4)

$$I_1Q = (I - I_1)S$$

Dividing (3) by (4):

BALANCE CONDITION OF THE WHEATSTONE BRIDGE

$$\frac{P}{Q} = \frac{R}{S}$$